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TENTAMEN RELATIVISTIC QUANTUM MECHANICS

Tuesday 24-06-2008, 09.00-12.00

On the first sheet write your name, address and student number. Write your name on all other sheets.

This examination consists of four problems, with in total 18 parts. The 18 parts carry equal weight in determining the final result of this examination.

$\hbar = c = 1$. The standard representation of the 4×4 Dirac gamma-matrices is given by:

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}, \quad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}.$$

PROBLEM 1

The Dirac equation is given by

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0.$$

The orbital angular momentum operator is $\vec{L} = \vec{x} \times \vec{p}$. The operators $\vec{p} = (p^1, p^2, p^3)$ are represented by $p^k = -i\partial/\partial x^k$. The spin angular momentum is given by $\vec{S} = \frac{1}{2}\gamma^5\gamma^0\vec{\gamma}$. The Hamiltonian H of the Dirac equation is

$$H = \gamma^0 \vec{\gamma} \cdot \vec{p} + m\gamma^0.$$

1.1 Evaluate $[\vec{L},\ \vec{\gamma} \cdot \vec{p}]$.

1.2 Show that $[\vec{L},\ H] = i\gamma^0(\vec{\gamma} \times \vec{p})$.

1.3 Show that $[\vec{S}, \gamma^0] = 0$.

?+ 1.4 Show that $[\vec{L} + \vec{S},\ H] = 0$.

PROBLEM 2

In the quantisation of the electromagnetic field one starts with the Lagrangian density

$$\mathcal{L}_1 = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu}, \tag{2.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

?+ 2.1 The fields $A_\mu(x)$ are taken to be the canonical coordinates. Determine the corresponding canonical momenta. Why are these coordinates and momenta inconvenient for the standard procedure of canonical quantisation?

2.2 Argue that an arbitrary field $A_\mu(x)$ is physically equivalent to another field $A'_\mu(x)$ which satisfies

$$\partial_\mu A^{\mu\prime} = 0.$$

2.3 Now we modify the Lagrangian density by including an additional term:

$$\mathcal{L}_2 = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - \tfrac{1}{2}(\partial_\mu A^\mu)^2. \tag{2.2}$$

Determine the canonical momenta for the Lagrangian density (2.2).

2.4 Derive the equation of motion for the field $A_\mu(x)$ from the Lagrangian density (2.2).

2.5 To make the quantum theory which follows from (2.2) equivalent to ordinary electrodynamics, which follows from (2.1), we have to somehow set $\partial_\mu A^\mu$ equal to zero. Explain, without going into extensive mathematical detail, how this should be done.

PROBLEM 3

A solution of the Dirac equation for a free spinor field is of the form

$$\psi(x) = u(p,s) e^{-ipx} \Big|_{p^0 = \omega_p}.$$

In this problem the momentum p is always on-shell, $p^0 = \omega_p$.

3.1 Show that $\psi(x)$ is also a solution of the Klein-Gordon equation.

3.2 The Dirac equation implies an equation for $u(p,s)$. Determine this equation.

? 3.3 Show that the matrix γ^0 has two eigenvalues $+1$ and two eigenvalues -1 (do not use an explicit representation for the γ-matrices!).

3.4 The solution for $u(p,s)$ is

$$u(p,s) = \frac{\gamma^\mu p_\mu + m}{\sqrt{2m(m+\omega_p)}} u(0,s)\,, \tag{3.1}$$

with $s = 1, 2$, and

$$\gamma^0 u(0,s) = u(0,s)\,. \tag{3.2}$$

Show that eq. (3.2) follows from eq. (3.1) in the restframe.

3.5 Show that

$$\bar{u}(p,s)u(p,t) = \delta_{st}\,.$$

PROBLEM 4

In this problem we consider interactions between electrons and photons.

4.1 Show that the annihilation of an electron and a positron into a single, free, photon is kinematically not allowed.

4.2 Show that the emission of a photon by a single, free, electron is kinematically forbidden.

not of interest

4.3 Draw two Feynman diagrams that contribute to the annihilation of an electron-positron pair into two photons at order e^2.

4.4 A muon is a particle that is in every respect (charge, spin,..) like the electron, the only difference is that it is heavier than the electron. Show that the decay of a muon into an electron and a photon is kinematically allowed. What is the energy of the electron in terms of the masses of electron and muon when the decay takes place in the muon rest frame?